FALL 2024 MATH 147 : MIDTERM EXAM I SOLUTIONS

When applicable, show all work to receive full credit. When in doubt, it is better to show more work than less. Each problem is worth 30 points.

Please work each problem on a separate sheet of paper, using the reverse side if necessary. Be sure to put your name on each page of your solutions. Good luck on the exam!

1. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a function of several variables.
   (i) When $n = 2$, define what it means for $f(x, y)$ to be differentiable at (a, b).
   (ii) Define what it means for $f(x_1, \dots, x_n)$ to be differentiable at $P = (a_1, \dots, a_n)$.
   (ii) Use the limit definition to show that $f(x, y,) = xy + 3x + 2y - 4$ is differentiable at $(0, 0)$.

Solution. (i) $f(x, y)$ is differentiable at (a, b) if $f_x(a, b), f_y(a, b)$ exist and $\lim_{(x,y)\to(a,b)} \frac{f(x,y)-L(x,y)}{||(y,y)-(a,b)||} = 0$, where $L(x, y) = f_x(a, b)(x - a) + f_y(a, b)(y - b) + f(a, b)$.

(ii) $f(x_1, \dots, x_n)$ is differentiable at P if each $f_{x_i}(P)$ exists and $\lim_{(x_1,\dots,x_n)\to P} \frac{F(x_1,\dots,x_n)-L(x_1,\dots,x_n)}{||(x_1,\dots,x_n)-P||} = 0$, where $L(x_1, \dots, x_n) = f_{x_1}(P)(x_1 - a_1) + \cdots + f_{x_n}(P)(x_n - a_n) + f(P)$.

(iii) An easy calculation give $L(x, y) = 3x + 2y - 4$. Calculating:

$$\begin{aligned}\lim_{(x,y)\to(0,0)} \frac{\{xy + 3x + 2y - 4\} - \{3x + 2y - 4\}}{\sqrt{x^2 + y^2}} &= \lim_{(x,y)\to(0,0} \frac{xy}{\sqrt{x^2 + y^2}} \\ &= \lim_{r\to 0} \frac{(r\cos(\theta))(r\sin(\theta)}{r} \quad \text{(for fixed } \theta) \\ &= \lim_{r\to 0} r\cos(\theta)\sin(\theta) \quad \text{(for fixed } \theta) \\ &= 0,\end{aligned}$$

so $f(x, y)$ is differentiable at (0,0).

2. True or False. Explain why you chose true or false.
   (i) For $f : \mathbb{R}^2 \to \mathbb{R}$, if every directional derivative of $f(x, y)$ exists at (a, b), then $f(x, y)$ is differentiable at (a, b). False, by Problem 7 on the review sheet.
   (ii) There exists $f : \mathbb{R}^2 \to \mathbb{R}$ such that the first and second order partial derivatives of $f(x, y)$ are continuous and $f_x = y$ and $f_y = -x$. False. The conditions on the partials imply $f_{xy} = f_{yx}$, but for the given function, $f_{xy} = 1$ and $f_{yx} = -1$.
   (iii) $f(x, y) = (x^2 + y^2, |y|)$ is differentiable at $(0, 1)$. True, since $x^2 + y^2$ and $|y|$ are both differentiable at (0,1).
   (iv) Let $f(x, y, z) = (5, 6, 7)$ for all $(x, y, z) \in \mathbb{R}^3$. Then $Df(\underline{a}) = [0\ 0\ 0]$, for all $\underline{a} \in \mathbb{R}^3$. False. Because $f : \mathbb{R}^3 \to \mathbb{R}^3, Df(5, 6, 7)$ is a 3×3 consisting entirely of zeros. No answers were marked wrong on this problem.
   (v) If $\vec{v} = v_1\vec{i} + v_2\vec{j}$ is a vector in $\mathbb{R}^2$, then the directional derivative of $f(x, y)$ at (a, b) in the direction of $\vec{v}$ is given by $v_1 f_x(a, b) + v_2 f_y(a, b)$. False, $\vec{v}$ must be a unit vector.

3. On this problem, you need to calculate carefully as your answer in the later parts may depend on what you have calculated in the earlier parts. For the function $f(x, y) = \begin{cases} \frac{x^4+x^2y^2+y^4}{x^2+y^2}, & \text{if } (x, y) \neq (0, 0) \\ 0, & \text{if } (x, y) = (0, 0) \end{cases}$

   (i) Determine if $f(x, y)$ is continuous at (0,0).
   (ii) Find formulas for $f_x(x, y)$ and $f_y(x, y)$ valid for all (x, y) in $\mathbb{R}^2$.
   (iii) Determine if $f(x, y)$ is differentiable at (0,0).
   (iv) Determine if $f_x(x, y)$ and $f_y(x, y)$ are continuous at (0,0) and explain the relevance of this to your answer in part (iii).
   (vi) Determine if $f_{xy}(0, 0) = f_{yx}(0, 0)$.

Solution. (i) For continuity, we have

$$\lim_{(x,y)\to(0,0)} f(x, y) = \lim_{r\to 0} \frac{r^4(\cos^4(\theta) + \cos^2(\theta)\sin^2(\theta) + \sin^4(\theta)}{r^2} = 0 = f(0, 0),$$

so $f(x,y)$ is continuous at (0,0).

(ii) For $(x,y) \neq (0,0)$, we can take partials in the usual way. Differentiating, we get $f_x(x,y) = \frac{2x^5+4x^3y^2}{(x^2+y^2)^2}$ and $f_y(x,y) = \frac{2y^5+4x^2y^3}{(x^2+y^2)^2}$. We must use limits to evaluate these partials at (0,0).

$$f_x(0,0) = \lim_{h\to 0} \frac{(0+h)^4 + (0+h)^2 \cdot 0 + 0^2}{h^2} = \lim_{h\to 0} h^2 = 0.$$

Similarly, $f_y(0,0) = 0$. Thus,

$$f_x(x,y) = \begin{cases} \frac{2x^5+4x^3y^2}{(x^2+y^2)^2}, \text{ if } (x,y) \neq (0,0) \\ 0, \text{ if } (x,y) = (0,0) \end{cases} \qquad f_y(x,y) = \begin{cases} \frac{2y^5+4x^2y^3}{(x^2+y^2)^2}, \text{ if } (x,y) \neq 0 \\ 0, \text{ if } (x,y) = (0,0). \end{cases}$$

(iii) At the point (0,0), $L(x,y) = 0 \cdot (x-0) + 0 \cdot (y-0) + 0 = 0$. Thus

$$\begin{aligned} \lim_{(x,y)\to(0,0)} \frac{f(x,y) - L(x,y)}{||(x,y)-(0,0)||} &= \lim_{(x,y)\to(0,0} \frac{\frac{x^4+x^2y^2+y^4}{x^2+y^2}}{\sqrt{x^2+y^2}} \\ &= \lim_{(x,y)\to(0,0)} \frac{x^4+x^2y^2+y^4}{(x^2+y^2)^{\frac{3}{2}}} \\ &= \lim_{r\to 0} \frac{r^4(\cos^4(\theta) + \cos^2(\theta)\sin^2(\theta) + \sin^4(\theta)}{r^3} \\ &= \lim_{r\to 0} r(\cos^4(\theta) + \cos^2(\theta)\sin^2(\theta) + \sin^4(\theta)) \\ &= 0, \end{aligned}$$

so $f(x,y)$ is differentiable at (0,0).

(iv) For continuity of $f_x(x,y)$ at (0,0), we check

$$\lim_{(x,y)\to(0,0)} \frac{2x^5+4x^3y^2}{(x^2+y^2)^2} = \lim_{r\to 0} \frac{2(r\cos(\theta))^5 + 4(r\cos(\theta))^3(r\sin(\theta))^2}{r^4} = \lim_{r\to 0} r \cdot (2\cos^5(\theta) + 4\cos^3(\theta)\sin^2(\theta)) = 0,$$

which shows that $f_x(x,y)$ is continuous at (0,0). The argument is exactly the same for $f_y(x,y)$, except the roles of x and y are reversed. The relevance of this to part (iii) is that equality of first partials in a neighborhood of (0,0) implies dfferentiability of $f(x,y)$ at (0,0). Since the partials of $f(x,y)$ are continuous everywhere, $f(x,y)$ is also differentiable everywhere, and in particular, at (0,0).

(v) We check $f_{xy}(0,0)$ using the limit definition and our formula for $f_x(x,y)$ form part (ii):

$$f_x(0,0) = \lim_{h\to 0} \frac{f_x(0,0+h) - f_x(0,0)}{h} = \lim_{h\to 0} \frac{\frac{20^5+40^3h^2}{(0^2+h^2)^2} - 0}{h} = \lim_{h\to 0} \frac{\frac{0}{h^4} - 0}{h} = 0.$$

The same calculation shows that $f_{yx}(0,0) = 0$, so that $f_{xy}(0,0) = f_{yx}(0,0)$.

Important Remark. The theorem on equality of mixed partials states that continuity of mixed partials implies equality of mixed partials. The function above shows that the converse of this theorem is **Not True**. That is, equality of mixed partials does not imply continuity of mixed partials. One can show that, for the function in Problem 3, neither $f_{xy}(x,y)$ nor $f_{yx}(x,y)$ are continuous at (0,0), even though $f_{xy}(0,0) = f_{yx}(0,0)$.

4. Heron's formula for the area of a triangle with sides having lengths x, y, z is given by

$$A(x,y,z) = \sqrt{s(s-x)(s-y)(s-z)},$$

where $s = \frac{1}{2}(x+y+z)$. Show that for a **fixed perimeter** P, the triangle with largest area is equilateral. Justify (in one way or another) that what you have found is, indeed, a maximum. Note: $P = 2s$. Hint: It suffices to maximize A^2. You may use the fact that the sum of any two sides of a triangle is greater than the remaining side.

Solution. It suffices to maximize A^2. Setting $z = P - x - y$ and $f(x,y) = A^2$, we have

$$\begin{aligned} f(x,y) &= \frac{P}{2}(\frac{P}{2} - 2)(\frac{P}{2} - y)(\frac{P}{2} - (P - x - y)) \\ &= \frac{P}{2}(\frac{P}{2} - x)(\frac{P}{2} - y)(x + y - \frac{P}{2}). \end{aligned}$$

Setting the partials equal to zero, we have:

$$f_x = \frac{P}{2}(-1)(\frac{P}{2} - y)(x + y - \frac{P}{2}) + \frac{P}{2}(\frac{P}{2} - x)(\frac{P}{2} - y) = 0 \qquad (*)$$

and

$$f_y = \frac{P}{2}(-1)(\frac{P}{2} - x)(x + y - \frac{P}{2}) + \frac{P}{2}(\frac{P}{2} - x)(\frac{P}{2} - y) = 0. \qquad (**)$$

We may divide (*) by $\frac{P}{2}$ and $(\frac{P}{2} - y)$ and (**) by $\frac{P}{2}$ and $(\frac{P}{2} - x)$, since $P \neq 0$ and neither x nor y can equal $\frac{P}{2}$, since the sum of any two sides of a triangle must be greater than the remaining side. Upon doing so, the system of equations becomes

$$-(x + y - \frac{P}{2}) + (\frac{P}{2} - x) = 0$$

$$-(x + y - \frac{P}{2}) + (\frac{P}{2} - y) = 0.$$

Subtracting the second equation from the first we get $(\frac{P}{2} - x) - (\frac{P}{2} - y) = 0$, which shows $x = y$. Substituting this into the first equation above gives $-(x + x - \frac{P}{2}) + (\frac{P}{2} - x) = 0$, which gives $x = \frac{P}{3}$. Thus, $y = \frac{P}{3}$ and since $z = P - x - y$, we get $z = \frac{P}{3}$, showing that we have an equilateral triangle. Why is this a maximum value? One could check using the second derivative test. Alternately, $A(x, y, z)$ is defined over the closed and bounded region of points in $\mathbb{R}^3$ satisfying $0 \leq x + y + z \leq P$, so the area obtained by taking $x = y = z = \frac{P}{3}$ is either an absolute maximum or absolute minimum. The area in this case is $\frac{\sqrt{3}P^2}{36}$. One can check that by taking the right triangle with sides $\frac{P}{4}$, $\frac{5P}{24}$ and hypotenuse $\frac{13P}{24}$, the resulting area $\frac{5P^2}{192}$ is less than $\frac{\sqrt{3}P^2}{36}$, so the area in question must be an absolute maximum.

5. Short Answer :

(i) Given $f(x, y)$, what condition guarantees that $f_{xy}(a, b) = f_{yx}(a, b)$?

(ii) If $f(x, y) = e^{xy}$, and $x = x(t), y = y(t)$, find a formula for $f'(t)$ as a function of t.

(iii) Let S denote the sphere of radius R centered at the origin, and assume (a, b, c) is a point on S. Find the equation of the plane tangent to S at (a, b, c)

(iv) Find the absolute maximum and absolute minimum of the function $f(x, y) = \ln(x^2 + y^2 + 1)$ on the closed disk $D = \{(x, y) \mid 0 \leq x^2 + y^2 \leq 10\}$. Hint: To calculate, or not calculate, that is the question.

(v) What is the best linear approximation to the function to the function $f(x, y) = \frac{1}{x^2+y^2+1}$ at (1,-1)? What does one need to check in order to confirm that the answer is indeed a good linear approximation?

Solution. (i) If $f_{xy}(x, y)$ and $f_{yx}(x, y)$ are continuous in an open disk about (a, b), then $f_{xy}(a, b) = f_{yx}(a, b)$.

(ii) $f'(t) = ye^{xy}x'(t) + xe^{xy}y'(t) = \{x'(t)y(t) + x(t)y'(t)\}e^{x(t)y(t)}$.

(iii) S is the level surface $f(x, y, z) = R^2$, for $f(x, y, z) = x^2 + y^2 + z^2$. Thus, the normal vector at the given point is $\nabla f(a, b, c) = 2a\vec{i} + 2b\vec{j} + 2c\vec{k}$. The tangent plane is given by $2a(x - a) + 2b(y - b) + 2c(z - c) = 0$.

(iv) The function $x^2 + y^2 + 1$ takes increasing constant values on circles centered at (0,0) with increasing radii, so that its maximum value 11 is obtained on the boundary of D. Its minimum value 1 is obtained at (0,0). Since ln is an increasing function, the maximum value of $f(x, y)$ is ln(11), and the minimum value is $\ln(1) = 0$.

(v) $f_x(x, y) = \frac{-2x}{(x^2+y^2+1)^2}$ and $f_y(x, y) = \frac{-2y}{(x^2+y^2+1)^2}$, so that $f_x(1, -1) = -\frac{2}{9}$ and $f_y(1, -1) = \frac{2}{9}$. Also: $f(1, -1) = \frac{1}{3}$. Taking $L(x, y) = -\frac{2}{9}(x - 1) + \frac{2}{9}(y + 1) + \frac{1}{3}$, $L(x, y)$ is the proposed best linear approximation. To confirm this, one must check that $f(x, y)$ is differentiable at (1, -1).

Optional Bonus Problems. Bonus problems must be completely correct to receive any bonus points.

(a) Let S denote the paraboloid given by the equation $z = f(x, y) = x^2 + y^2$. Let C denote the curve on S consisting of the points on S lying above the line $y = 2x + 3$. Find the equation of the line tangent to C at the point $(1, 5, 26)$. (10 points)

Solution. For (a): Since (0,3) and (1,5) lie on the line, we may take $(1, 2)$ as a direction vector for this line, so that the parametric equation for the given line is $h(t) = (1, 5) + t(1, 2) = (1 + t, 5 + 2t)$. On the other hand, $f_x(1, 5) = 2$ $f_y(1, 5) = 10$, so $z = 2(x - 1) + 10(y - 5) + 26 = 2x + 10y - 26$ is the tangent plane to the surface at the given point. Lifting the line $h(t)$ to this plane gives the corresponding tangent line

$$L(t) = (1 + t, 5 + 2t, 2(1 + t) + 10(5 + 2t) - 26) = (1 + t, 5 + 2t, 26 + 22t) = (1, 5, 26) + t(1, 2, 22).$$

Important Observation. The direction vector for the tangent line is

$$(1, 2, 22) = (1, 2, 1 \cdot f_x(1, 5) + 2 \cdot f_y(1, 5)) = (v_1, v_2, \vec{v} \cdot \nabla f(1, 5)),$$

where $\vec{v} = v_1\vec{i} + v_2\vec{j}$ is the given direction vector for the line $y = 2x + 3$. In general, suppose S is the surface given by the graph of $z = f(x, y)$ and $(a, b, f(a, b))$ is a point on S. Let L_0 be a line in $\mathbb{R}^2$ passing through (a, b) and C

the curve on S consisting of points on S lying over L. Then the parametric equation of the line tangent to C at the point $(a, b, f(a, b))$ is given by

$$L(t) = (a, b, f(a, b)) + t(u_1, u_2, D_{\vec{u}} f(a, b)),$$

where $\vec{u} = u_1 \vec{i} + u_2 \vec{j}$ is a unit direction vector for the line L_0 in $\mathbb{R}^2$ passing through (a, b).

For (b): We have

$$\begin{aligned}
\nabla(fg) &= (fg)_x \vec{i} + (fg)_y \vec{j} + (fg)_z \vec{k} \\
&= (f_x g + f g_x)\vec{i} + (f_y g + f g_y)\vec{j} + (f_z g + f g_z)\vec{k} \\
&= (f g_x \vec{i} + f g_y \vec{j} + f g_z \vec{k}) + (g f_x \vec{i} + g f_y \vec{j} + g f_z \vec{k}) \\
&= f(g_x \vec{i} + g_y \vec{j} + g_z \vec{j}) + g(f_x \vec{i} + f_y \vec{j} + f_z \vec{k}) \\
&= f \nabla g + g \nabla f.
\end{aligned}$$